

Exercise 1

A spring has natural length 0.75 m and a 5-kg mass. A force of 25 N is needed to keep the spring stretched to a length of 1 m. If the spring is stretched to a length of 1.1 m and then released with velocity 0, find the position of the mass after t seconds.

Solution

In order to determine the spring constant, use the fact that 25 N is needed to stretch the spring to 1 m.

$$F = k(x - x_0)$$

$$25 \text{ N} = k(1 - 0.75) \text{ m}$$

$$k = 100 \frac{\text{N}}{\text{m}}$$

Apply Newton's second law to obtain the equation of motion.

$$F = ma$$

Use the fact that acceleration is the second derivative of position $a = d^2x/dt^2$ and the fact that the spring force $F = -kx$ is the only force acting on the mass.

$$-kx = m \frac{d^2x}{dt^2}$$

Divide both sides by m .

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x = e^{rt} \rightarrow \frac{dx}{dt} = re^{rt} \rightarrow \frac{d^2x}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$r^2e^{rt} = -\frac{k}{m}e^{rt}$$

Divide both sides by e^{rt} .

$$r^2 = -\frac{k}{m}$$

Solve for r .

$$r = \left\{ -i\sqrt{\frac{k}{m}}, i\sqrt{\frac{k}{m}} \right\}$$

Two solutions to the ODE are $e^{-i\sqrt{k/m}t}$ and $e^{i\sqrt{k/m}t}$.

By the principle of superposition, then, the general solution to equation (1) is

$$\begin{aligned} x(t) &= C_1 e^{-i\sqrt{k/m}t} + C_2 e^{i\sqrt{k/m}t} \\ &= C_1 \left(\cos \sqrt{\frac{k}{m}}t - i \sin \sqrt{\frac{k}{m}}t \right) + C_2 \left(\cos \sqrt{\frac{k}{m}}t + i \sin \sqrt{\frac{k}{m}}t \right) \\ &= (C_1 + C_2) \cos \sqrt{\frac{k}{m}}t + (-iC_1 + iC_2) \sin \sqrt{\frac{k}{m}}t \\ &= C_3 \cos \sqrt{\frac{k}{m}}t + C_4 \sin \sqrt{\frac{k}{m}}t, \end{aligned}$$

where C_3 and C_4 are arbitrary constants. Differentiate it with respect to t to get the velocity.

$$\frac{dx}{dt} = -C_3 \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}t + C_4 \sqrt{\frac{k}{m}} \cos \sqrt{\frac{k}{m}}t$$

Apply the initial conditions, $x(0) = 1.1 - 0.75 = 0.35$ and $x'(0) = 0$, to determine C_3 and C_4 .

$$x(0) = C_3 = 0.35$$

$$\frac{dx}{dt}(0) = C_4 \sqrt{\frac{k}{m}} = 0$$

Solving this system of equations yields $C_3 = 0.35$ and $C_4 = 0$.

$$x(t) = 0.35 \cos \sqrt{\frac{k}{m}}t$$

Therefore, plugging in $m = 5$ kg and $k = 100$ N/m, the displacement from equilibrium is

$$x(t) = 0.35 \cos \sqrt{20}t.$$

